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Four Aspects of the Mathematical Theory of Economic Equilibrium

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The observed state of an economy can be viewed as an equilibrium resulting from the interaction of a large number of agents with partially conflicting interests. Taking this viewpoint, exactly one hundred years ago, Léon Walras presented in his *Eléments d'Economie Politique Pure* the first general mathematical analysis of this equilibrium problem. During the last four decades, Walrasian theory has given rise to several developments that required the use of basic concepts and results borrowed from diverse branches of mathematics. In this article, I propose to review four of them.

1. The existence of economic equilibria. As soon as an equilibrium state is defined for a model of an economy, the fundamental question of its existence is raised. The first solution of this problem was provided by A. Wald [**1933–1935**], and after a twenty-year interruption, research by a large number of authors has steadily extended the framework in which the existence of an equilibrium can be established. Although no work was done on the problem of existence of a Walrasian equilibrium from the early thirties to the early fifties, several contributions, which, later on, were to play a major role in the study of that problem, were made in related areas during that period. One of them was a lemma proved by J. von Neumann [**1937**] in connection with his model of economic growth. This lemma was reformulated by S. Kakutani [**1941**] as a fixed-point theorem which became the most powerful tool for proofs of existence in economics. Another contribution, due to J. Nash [**1950**], was the first use of that tool in the solution of a problem of socia

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equilibrium. For later reference we state Kakutani's theorem. Given two sets U and V, a *correspondence* ρ from U to V associates with every element $u \in U$, a nonempty subset $\rho(u)$ of V.

THEOREM. *If D is a nonempty, compact, convex subset of a Euclidean space, and ρ is a convex-valued, closed-graph correspondence from D to D, then there is d^* such that $d^* \in \rho(d^*)$.*

As a simple prototype of a Walrasian equilibrium problem, we now consider an exchange economy with l commodities, and a finite set A of consumers. The consumption of consumer $a \in A$ is described by a point x_a in R^l_+; the ith coordinate x^i_a of x_a being the quantity of the ith commodity that he consumes. A price system p is an l-list of strictly positive numbers, i.e., a point in $P = \text{Int } R^l_+$; the ith coordinate of p being the amount to be paid for one unit of the ith commodity. Thus the value of x_a relative to p is the inner product $p \cdot x_a$. Given the price vector $p \in P$, and his wealth $w \in L$, the set of strictly positive numbers, consumer a is constrained to satisfy the budget inequality $p \cdot x_a \leqq w$. Since multiplication of p and w by a strictly positive number has no effect on the behavior of consumers, we can normalize p, restricting it to the strictly positive part of the unit sphere $S = \{p \in P \mid \|p\| = 1\}$. We postulate that, presented with the pair $(p, w) \in S \times L$, consumer a demands the consumption vector $f_a(p, w)$ in R^l_+, and that the demand function f_a is continuous. If that consumer is insatiable, f_a also satisfies

$$(1) \qquad \textit{for every } (p, w) \in S \times L, \quad p \cdot f_a(p, w) = w.$$

To complete the description of the economy $\mathscr{E}$, we specify for consumer a an initial endowment vector $e_a \in P$. Thus the characteristics of consumer a are the pair (f_a, e_a), and $\mathscr{E}$ is the list $((f_a, e_a))_{a \in A}$ of those pairs for $a \in A$. Consider now a price vector $p \in S$. The corresponding wealth of consumer a is $p \cdot e_a$; his demand is $f_a(p, p \cdot e_a)$. Therefore the excess demand of the economy is

$$F(p) = \textstyle\sum_{a \in A} [f_a(p, p \cdot e_a) - e_a].$$

And p is an equilibrium price vector if and only if $F(p) = 0$. Because of (1), the function F from S to R^l satisfies

Walras'law. $p \cdot F(p) = 0$.

Consequently, F is a continuous vector field on S, all of whose coordinates are bounded below. Finally, we make an assumption about the behavior of F near ∂S.

Boundary condition. If p_n in S tends to p_0 in ∂S, then $\{F(p_n)\}$ is unbounded.

This condition expresses that every commodity is collectively desired. Here and below I freely make unnecessarily strong assumptions when they facilitate the exposition. Of the many variants of the existence theorem that have been proposed, I select the following statement by E. Dierker [**1974**, §8], some of whose antecedents were L. McKenzie [**1954**], D. Gale [**1955**], H. Nikaido [**1956**], and K. Arrow and F. Hahn [**1971**].

THEOREM. *If F is continuous, bounded below, and satisfies Walras' law and the boundary condition, then there is an equilibrium.*

We indicate the main ideas of a proof because they will recur in this section and in the next. Here it is most convenient to normalize the price vector so that it belongs to the simplex $\Pi = \{p \in R^l_+ \mid \sum_{i=1}^l p^i = 1\}$.

Consider a price vector $p \notin \partial\Pi$ yielding an excess demand $F(p) \neq 0$. According to a commonly held view of the role of prices, a natural reaction of a price-setting agency to this disequilibrium situation would be to select a new price vector so as to make the excess demand $F(p)$ as expensive as possible, i.e., to select (K. Arrow and G. Debreu [1954]) a price vector in the set

$$\mu(p) = \left\{\pi \in \Pi \,\middle|\, \pi \cdot F(p) = \operatorname*{Max}_{q \in \Pi} q \cdot F(p)\right\}.$$

When $p \in \partial\Pi$, the excess demand is not defined. In this case, we let $\mu(p) = \{\pi \in \Pi \mid \pi \cdot p = 0\}$.

By Kakutani's theorem, the correspondence μ from Π to Π has a fixed point p^*. Obviously, $p^* \notin \partial\Pi$. But then $p^* \in \mu(p^*)$ implies $F(p^*) = 0$.

From the fact that $\mu(p)$ is always a face of Π one suspects (rightly as we will see in the next section) that Kakutani's theorem is too powerful a tool for this result. But such is not the case in the general situation to which we will turn after having pointed out the broad interpretation that the concept of commodity must be given. In contemporary Walrasian theory, a commodity is defined as a good or a service with specified physical characteristics, to be delivered at a specified date, at a specified location, if (K. Arrow [1953]) a specified event occurs. Aside from this mere question of interpretation of a concept, the model can be expanded so as to include a finite set B of producers. Producer $b \in B$ chooses a production vector y_b (whose positive coordinates correspond to outputs, and negative coordinates to inputs) in his production set Y_b, a nonempty subset of R^l, interpreted as the set of feasible production vectors. When the price vector p is given, producer b actually chooses his production vector in a nonempty subset $\psi_b(p)$ of Y_b. It is essential here, as it was not in the case of consumers, to provide for situations in which p does not uniquely determine the reaction of every producer, which may arise for instance if producer b maximizes his profit $p \cdot y_b$ in a cone Y_b with vertex 0 (constant returns to scale technology). In an economy with production, consumer a not only demands goods and services, but also supplies certain quantities of certain types of labor, which will appear as negative coordinates of his consumption vector x_a; this vector x_a is constrained to belong to his consumption set X_a, a given nonempty subset of R^l. A suitable extension of the concept of demand function covers this case. However, the wealth of a consumer is now the sum of the value of his endowment vector and of his shares of the profits of producers. In this manner, an integrated model of consumption and production is obtained, in which a state of the economy is a list $((x_a)_{a \in A}, (y_b)_{b \in B}, p)$ of vectors of R^l, where, for every $a \in A$, $x_a \in X_a$; for every $b \in B$, $y_b \in Y_b$; and $p \in \Pi$. The problem of existence of an equilibrium for such an economy has often been reduced to a situation similar to that of the last theorem, the continuous excess demand function being replaced by an excess demand correspondence with a closed graph. Alternatively, it can be formulated in the following general terms, in the spirit of J. Nash [1950]. The social system is composed of a finite set C of agents. For each $c \in C$, a set D_c of possible actions is given. Consequently, a state of the system is an element d of the product $D = \times_{c \in C} D_c$. We denote by $d_{C \backslash c}$ the list of actions obtained by deleting d_c from d. Given $d_{C \backslash c}$, i.e., the actions chosen by all the other agents, agent c reacts by choosing his own action in the set $\rho_c(d_{C \backslash c})$. The state d^* is an equilibrium if and only if, for every $c \in C$, $d_c^* \in \rho_c(d_{C \backslash c}^*)$. Thus, the reaction correspondence ρ from D to D being defined by $\rho(d) = \times_{c \in C} \rho_c(d_{C/c})$, the state d^* is an equilibrium if and only if it is a fixed point of ρ. In the integrated economic model of consumption and production that we discussed, one of the agents is the impersonal market to which we assign the reaction correspondence μ introduced in the proof of the existence theorem.

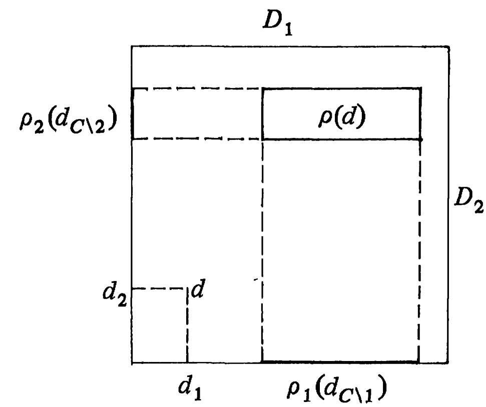

FIGURE 1

Still broader interpretations and further extensions of the preceding model have been proposed. They include negative or zero prices, preference relations with weak properties instead of demand functions for consumers, measure spaces of agents, infinite-dimensional commodity spaces, monopolistic competition, public goods, redistribution of income, indivisible commodities, transaction costs, money, the use of nonstandard analysis,.... Since this extensive, and still rapidly growing, literature cannot be surveyed in detail here, I refer to the excellent account by K. Arrow and F. Hahn **[1971]**, to the books mentioned in the next sections, and to recent volumes of Econometrica, Journal of Economic Theory, and Journal of Mathematical Economics.

2. The computation of economic equilibria. While the first proof of existence is forty years old, decisive steps towards an efficient algorithm for the computation of Walras equilibria were taken only during the last decade. In 1964, C. Lemke and J. Howson gave an effective procedure for the computation of an equilibrium of a non-zero-sum two-person game. H. Scarf **[1967]**, **[1973]** then showed how a technique similar to that of C. Lemke and J. Howson could be used to compute an approximate Walras equilibrium, and proposed a general algorithm for the calculation of an approximate fixed point of a correspondence. This algorithm, which has revealed itself to be surprisingly efficient, had the drawback of not permitting a gradual improvement of the degree of approximation of the solution. An essential extension due to C. Eaves [**1972**], [**1974**], stimulated by a fixed-point theorem of F. Browder [**1960**], overcame this difficulty.

Before presenting a version of the algorithm based on H. Scarf [**1973**], and C. Eaves [**1974**], we note that in the preceding proof of existence, we have actually associated with every point $p \in \Pi$ a set $\Lambda(p)$ of integers in $I = \{1, \cdots, l\}$, as follows.

$$\begin{aligned} \Lambda(p) &= \{i \mid F^i(p) = \mathrm{Max}_j F^j(p)\} && \text{if } p \notin \partial\Pi, \\ &= \{i \mid p^i = 0\} && \text{if } p \in \partial\Pi. \end{aligned}$$

The point p^* is an equilibrium if and only if $\Lambda(p^*) = I$, in other words, if and only if it is in the intersection of the closed sets $E_i = \{p \mid i \in \Lambda(p)\}$. Showing that this intersection is not empty would yield an existence proof in the manner of D. Gale [**1955**].

We specify our terminology. By a simplex, we always mean a closed simplex, and, of course, similarly for a face of a simplex. *A facet* of an n-simplex is an $(n-1)$-face. For each $p \in \Pi$, select now a label $\lambda(p)$ in $\Lambda(p)$. A set M of points is said to be *completely labeled*, abbreviated to *c.l.*, if the set $\lambda(M)$ of its labels is I. The labeling λ is chosen so as to satisfy the following restrictions on $\partial\Pi$:

(α) the set of vertices of Π is c.l.,

(β) no facet of Π is c.l.[1]

The algorithm will yield a c.l. set of l points of Π whose diameter can be made arbitrarily small, and consequently a point of Π at which the value of F can be made arbitrarily small.

Let T be the part of R^l_+ that is above Π, and $\mathscr{T}$ be a standard regular triangulation

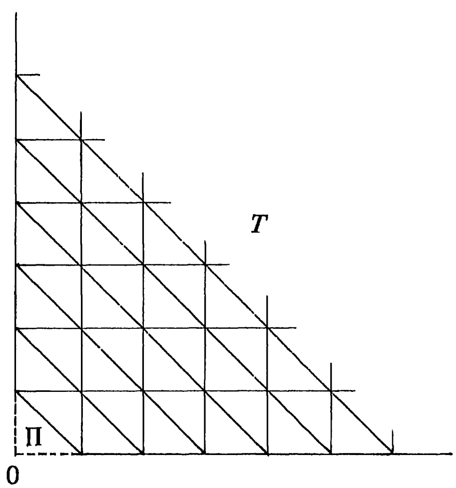

FIGURE 2

[1]Here is a simple example of a labeling of $\partial\Pi$ satisfying those restrictions. Given $p \in \partial\Pi$, select any $\lambda(p)$ in $\Lambda(p)$ such that $\lambda(p) - 1 \pmod{l}$ is not in $\Lambda(p)$.

of T having for vertices the points of T with integral coordinates, used by H. Kuhn **[1960]**, **[1968]**, T. Hansen **[1968]**, and C. Eaves **[1972]**, and illustrated by the figure. (Other considerably more efficient triangulations of, or more appropriately pseudo-manifold structures on, T have been used, C. Eaves **[1972]**, **[1974]**.) Give any point in T the same label as its projection from 0 into Π; and say that two $(l-1)$-simplexes of $\mathscr{T}$ are *adjacent* if there is an l-simplex of $\mathscr{T}$ of which they are facets. Consider now an $(l-1)$-simplex s of $\mathscr{T}$ with c.l. vertices.

(i) If $s = \Pi$, s is a facet of exactly one l-simplex of $\mathscr{T}$; hence there is exactly one $(l-1)$-simplex of $\mathscr{T}$ with c.l. vertices adjacent to s.

(ii) If $s \neq \Pi$, because of (β), s is not in the boundary of T; therefore s is a facet of exactly two l-simplexes of $\mathscr{T}$; hence there are exactly two $(l-1)$-simplexes of $\mathscr{T}$ with c.l. vertices adjacent to s.

The algorithm starts from $s^0 = \Pi$. Take s^1 to be the unique $(l-1)$-simplex of $\mathscr{T}$ with c.l. vertices adjacent to s^0. For $k > 0$, take s^{k+1} to be the unique $(l-1)$-simplex of $\mathscr{T}$ with c.l. vertices adjacent to s^k, and other than s^{k-1}. Clearly this algorithm never returns to a previously used $(l-1)$-simplex and never terminates. Given any integer n, after a finite number of steps, one obtains an $(l-1)$-simplex with c.l. vertices above the hyperplane $\{p \in R^l \mid \sum_{i=1}^{l} p^i = n\}$. Projecting from 0 into Π, one obtains a sequence of c.l. sets of l points of Π whose diameter tends to 0 as n tends to $+\infty$.

An approximate fixed point (i.e., a point close to its image) of a continuous function from a finite-dimensional, nonempty, compact, convex set to itself can be obtained by a direct application of this algorithm. But in order to solve the analogous problem for a fixed point of a correspondence, and consequently, for a Walras equilibrium of an economy with production, H. Scarf and C. Eaves have used vector labels rather than the preceding integer labels. With every point p of Π, one now associates a suitably chosen vector $\lambda(p)$ in R^{l-1}, and one says that a set M of points of Π is c.l. if the origin of R^{l-1} belongs to the convex hull of $\lambda(M)$. As before, the labeling λ of Π is restricted to satisfy (α) and (β). The last two paragraphs can then be repeated word for word with the following single exception. Let σ be an l-simplex of $\mathscr{T}$, and s be a facet of σ with c.l. vertices. Denote by V_σ (resp. V_s) the set of vertices of σ (resp. of s). If $\lambda(V_\sigma)$ is in general position in R^{l-1}, then 0 is interior to the convex hull of $\lambda(V_s)$, and there is exactly one other facet of σ with c.l. vertices. However, if $\lambda(V_\sigma)$ is not in general position, a degenerate case where there are several other facets of σ with c.l. vertices may arise. An appropriate use of the lexic refinement of linear programming resolves this degeneracy. In this general form, the algorithm can indeed be directly applied to the computation of approximate Kakutani fixed points.

The simplicity of this algorithm is very appealing, but its most remarkable feature is its efficiency. Experience with several thousand examples has been reported, in particular in H. Scarf **[1973]** and R. Wilmuth **[1973]**. As a typical case of the version of the integer-labeling algorithm presented above (which uses an inefficient triangulation of T), let $l = 10$. To reach an elevation $n = 100$ in T, i.e., a triangulation of Π for which every edge is divided into 100 equal intervals, the number of iterations required rarely exceeds 2,000, and the computing time on an IBM 370 is usually less than 15 seconds. The number of vertices that are examined in the computation is therefore a small fraction of the number of vertices of the triangulation of Π at elevation 100.

The best general reference on the problem discussed in this section is H. Scarf **[1973]**. *Mathematical Programming* is a good bibliographical source for more recent developments.

3. Regular differentiable economies. The model $\mathscr{E} = ((f_a, e_a))_{a \in A}$ of an exchange economy presented at the beginning of § 1 would provide a complete explanation of the observed state of that economy in the Walrasian framework if the set $E(\mathscr{E})$ of its equilibrium price vectors had exactly one element. However, this global uniqueness requirement has revealed itself to be excessively strong, and was replaced, in the last five years, by that of local uniqueness. Not only does one wish $E(\mathscr{E})$ to be discrete, one would also like the correspondence E to be continuous. Otherwise, the slightest error of observation on the data of the economy might lead to an entirely different set of predicted equilibria. This consideration, which is common in the study of physical systems, applies with even greater force to the study of social systems. Basic differential topology has provided simple and satisfactory answers to the two questions of discreteness of $E(\mathscr{E})$, and of continuity of E.

At first, we keep the list $f = (f_a)_{a \in A}$ of demand functions fixed, and we assume that each one of them is of class C^r ($r \geqq 1$). Thus an economy is identified with the point $e = (e_a)_{a \in A}$ in P^A. We denote by E the set of $(e, p) \in P^A \times S$ such that p is an equilibrium price vector for the economy e, and by $E(e)$ the set of equilibrium price vectors associated with a given e. The central importance of the manifold E, or of a related manifold of S. Smale **[1974]**, has been recognized by S. Smale **[1974]** and Y. Balasko **[1974a]**. Recently, Y. Balasko **[1974b]** has noticed the property of C^r-isomorphism to P^A.

THEOREM. *E is a C^r-submanifold of $P^A \times S$ of the same dimension as P^A. If for every $a \in A$ the range of f_a is contained in P, then E is C^r-isomorphic to P^A.*

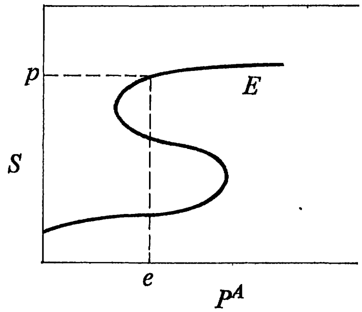

FIGURE 3

Now let π be the projection $P^A \times S \to P^A$, and $\bar{\pi}$ be its restriction to the manifold E.

DEFINITION. The economy $\mathscr{E} = (f, e)$ is *regular* if e is a regular value of $\tilde{\pi}$. It is *critical* if it is not regular.

By Sard's theorem, the set of critical e has Lebesgue measure zero. Suppose in addition we assume that every demand function f_a satisfies the

Strong boundary condition. If (p_n, w_n) in $S \times L$ tends to (p_0, w_0) in $\partial S \times L$, then $\{f_a(p_n, w_n)\}$ is unbounded.

Then we readily obtain that $\tilde{\pi}$ is proper (Y. Balasko **[1974b]**). In this case the critical set is closed (relative to P^A). It is therefore negligible in a strong sense. As for economies in the regular set $\mathscr{R}$, the complement of the critical set, they are well behaved in the following sense. At $e \in \mathscr{R}$, the compact set $E(e) = \tilde{\pi}^{-1}(e)$ is discrete, therefore finite, and $\tilde{\pi}^{-1}$ is locally a C^r-diffeomorphism.

In order to prepare for the discussion of regular economies in the context of the next section, we note an equivalent definition (E. and H. Dierker **[1972]**) of a critical point of the manifold E for $\tilde{\pi}$. Given e, let $F(p)$ be the excess demand associated with p, and denote by $\hat{F}(p)$ the projection of $F(p)$ into some fixed $(l - 1)$-dimensional coordinate subspace of R^l. Because of Walras' law, and because p is strictly positive, $F(p) = 0$ is equivalent to $\hat{F}(p) = 0$. Let then $J[\hat{F}(p)]$ be the Jacobian determinant of $\hat{F}$ at p. As Y. Balasko **[1974b]** shows, (e, p) is a critical point of $\tilde{\pi}$ if and only if $J[\hat{F}(p)] = 0$.

Since it is desirable to let demand functions vary as well as initial endowments (F. Delbaen **[1971]**, E. and H. Dierker **[1972]**), we endow the set D of C^r demand functions ($r \geqq 1$) satisfying the strong boundary condition with the topology of uniform C^r-convergence.

An economy $\mathscr{E}$ is now defined as an element of $(D \times P)^A$, a regular element of the latter space being a pair (f, e) for which the Jacobian determinant introduced in the last paragraph is different from zero for every equilibrium price vector associated with (f, e). The regular set is then shown to be open and dense in $(D \times P)^A$. Another extension, by S. Smale **[1974]**, established the same two properties of the regular set in the context of utility functions with weak properties, rather than in the context of demand functions.

Still further generalizations, for instance, to cases where production is possible, have been obtained. E. Dierker **[1974]** surveys a large part of the area covered in this section more leisurely than I did. Recent volumes of the three journals listed at the end of §1 are also relevant here.

4. The core of a large economy. So far the discussion of consumer behavior has been in terms of demand functions. We now introduce for consumer a the more basic concept of a binary preference relation $\precsim_a$ on R^l_+, for which we read "$x \precsim_a y$" as "for agent a, commodity vector y is at least as desired as commodity vector x." The relation of strict preference "$x \prec_a y$" is defined by "$x \precsim_a y$ and not $y \precsim_a x$," and of indifference "$x \sim_a y$" by "$x \precsim_a y$ and $y \precsim_a x$." Similary, for two vectors x, y in R^l we denote by "$x \leqq y$" the relation "$y - x \in R^l_+$," by "$x < y$" the relation "$x \leqq y$ and not $y \leqq x$," and by "$x \ll y$" the relation "$y - x \in P$."

We assume that $\precsim_a$ is a complete preorder with a closed graph, and that it satisfies the monotony condition, $x < y$ implies $x \prec_a y$, expressing the desirability of all commodities for consumer a. The set of preference relations satisfying these assumptions is denoted by $\mathscr{P}$, and viewing an element of $\mathscr{P}$ as a closed subset of R^{2l}, we endow $\mathscr{P}$ with Hausdorff's [1957] topology of closed convergence (Y. Kannai [1970]).

The characteristics of consumer $a \in A$ are now a pair $(\precsim_a, e_a)$ of a preference relation in $\mathscr{P}$, and an endowment vector in R^l_+. Thus an exchange *economy* $\mathscr{E}$ is a function from A to $\mathscr{P} \times R^l_+$. The result of any exchange process in this economy is an *allocation*, i.e., a function x from A to R^l_+, that is *attainable* in the sense that $\sum_{a\in A} x_a = \sum_{a\in A} e_a$.

A proposed allocation x is *blocked* by a coalition E of consumers if

(i) $E \neq \emptyset$,

and the members of E can reallocate their own endowments among themselves so as to make every member of E better off, i.e., if

(ii) there is an allocation y such that $\sum_{a\in E} y_a = \sum_{a\in E} e_a$ and, for every $a \in E$, $x_a \prec_a y_a$.

From this viewpoint, first taken by F. Edgeworth [1881], only the unblocked attainable allocations are viable. The set of those allocations is the *core* $C(\mathscr{E})$ of the economy. The goal of this section is to relate the core to the equilibrium concept that underlies the analysis of the first three sections. Formally, we define a *Walras allocation* as an attainable allocation x for which there is a price system $p \in \Pi$ such that, for every $a \in A$, x_a is a greatest element for $\precsim_a$ of the budget set $\{z \in R^l_+ \mid p\cdot z \leqq p\cdot e_a\}$.

The set of Walras allocations of $\mathscr{E}$ is denoted by $W(\mathscr{E})$. It satisfies the mathematically trivial but economically important relation $W(\mathscr{E}) \subset C(\mathscr{E})$.

Simple examples show that for small economies the second set is much larger than the first. However, F. Edgeworth [1881] perceived that as the number of agents tends to $+\infty$ in such a way that each one of them becomes insignificant relative to their totality, the two sets tend to coincide. The conditions under which F. Edgeworth proved his limit theorem were very special. The first generalization was obtained by H. Scarf [1962], after M. Shubik [1959] had called attention to the connection between F. Edgeworth's "contract curve" and the game-theoretical concept of the core. The problem was then placed in its natural setting by R. Aumann [1964]. The agents now form a positive measure space $(A, \mathscr{A}, \nu)$ such that $\nu(A) = 1$. The elements of $\mathscr{A}$ are the *coalitions*, and for $E \in \mathscr{A}$, $\nu(E)$ is interpreted as the fraction of the totality of agents in coalition E. Since the characteristics of an agent $a \in A$ are the pair $(\precsim_a, e_a)$, an *economy* $\mathscr{E}$ is defined (W. Hildenbrand [1974]), as a measurable function from A to $\mathscr{P} \times R^l_+$ such that e is integrable. The definitions of an unblocked attainable allocation and of a Walras allocation are extended in an obvious fashion. As trivially as before $W(\mathscr{E}) \subset C(\mathscr{E})$. But in the case in which the space of agents is atomless, i.e., in which every agent is negligible, R. Aumann [1964] has proved the

THEOREM. *If the economy $\mathscr{E}$ is atomless and $\int_A e\, d\nu \gg 0$, then $W(\mathscr{E}) = C(\mathscr{E})$.*

This remarkable result reconciles two fundamental and a priori very different equilibrium concepts. Its proof can be based (K. Vind [**1964**]) on Lyapunov's theorem on the convexity of the range of an atomless finite-dimensional vector measure.

There remains to determine the extent to which the equality of the core and of the set of Walras allocations holds approximately for a finite economy with a large number of nearly insignificant agents. This program is the object of W. Hildenbrand [**1974**], one of whose main results we now present.

Letting $K = \mathscr{P} \times R^l_+$ be the set of agents' characteristics, we introduce the basic concepts associated with the economy $\mathscr{E}$ that we need. The image measure $\mu = \nu \circ \mathscr{E}^{-1}$ of ν via $\mathscr{E}$ is a probability on K called the *characteristic distribution* of $\mathscr{E}$. Given an allocation x for $\mathscr{E}$ (i.e., an integrable function from A to R^l_+), consider the function γ_x from A to $K \times R^l_+$ defined by $\gamma_x(a) = (\mathscr{E}(a), x(a))$. The image measure $\nu \circ \gamma_x^{-1}$ of ν via γ_x is a probability on $K \times R^l_+$ called the *characteristic-consumption distribution* of x. We denote by $\mathscr{D}_W(\mathscr{E})$ the set of characteristic-consumption distributions of the Walras allocations of $\mathscr{E}$, and similarly by $\mathscr{D}_C(\mathscr{E})$ the set of characteristic-consumption distributions of the core allocations of $\mathscr{E}$. Finally, we formalize the idea of a competitive sequence of finite economies. $\# A_n$ will denote the number of agents of $\mathscr{E}_n$, μ_n the characteristic distribution of $\mathscr{E}_n$, and pr_2 the projection from K into R^l_+. The sequence $(\mathscr{E}_n)$ is *competitive* if

(i) $\# A_n \to +\infty$,

(ii) μ_n converges weakly to a limit μ,

(iii) $\int pr_2\, d\mu_n \to \int pr_2\, d\mu \gg 0$.

We denote by $\mathscr{E}^\mu$ the economy defined as the identity map from K, endowed with its Borel σ-field $\mathscr{B}(K)$, and the measure μ, to K. Then, endowing the set of probability measures on $K \times R^l_+$ with the topology of weak convergence, we obtain the theorem of W. Hildenbrand [**1974**, Chapter 3].

THEOREM. *If the sequence $(\mathscr{E}_n)$ is competitive, and U is a neighborhood of $\overline{\mathscr{D}_W(\mathscr{E}^\mu)}$, then, for n large enough, $\mathscr{D}_C(\mathscr{E}_n) \subset U$.*

To go further, and to obtain full continuity results, as well as results on the rate of convergence of the core of $\mathscr{E}_n$, we need an extension (F. Delbaen [**1971**], K. Hildenbrand [**1974**], and H. Dierker [**1974**]) of the concepts and of the propositions of § 3 to the present context of a measure space of agents. Specifically, we place ourselves in the framework of H. Dierker [**1974**]. In addition to being in $\mathscr{P}$, the preference relations of consumers are now assumed to satisfy the following conditions. For every point $x \in P$, the preference-or-indifference set $\{y \in P \mid x \precsim y\}$ is convex, and the indifference set $I(x) = \{y \in P \mid y \sim x\}$ is a C^2-hypersurface of P whose Gaussian curvature is everywhere nonzero, and whose closure relative to R^l is contained in P. Finally denoting by $g(x)$ the positive unit normal of $I(x)$ at the point x, we assume that g is C^1 on P. These conditions make it possible to identify the preference relation $\precsim$ with the C^1 vector field g on P.

The set G of these vector fields is endowed with the topology of uniform C^1 convergence on compact subsets. $\mathcal{M}$ then denotes the set of characteristic distributions on $G \times P$ with compact support. The assumptions that we have made imply that every agent has a C^1 demand function. Therefore it is possible to define a *regular* element μ of $\mathcal{M}$ as a characteristic distribution μ in $\mathcal{M}$ such that the Jacobian determinant introduced in §3 is different from zero for every equilibrium price vector associated with μ. Having suitably topologized the set $\mathcal{M}$, one can give, in the manner of H. Dierker **[1974]**, general conditions under which the regular set is open and dense in $\mathcal{M}$.

In this framework, the following result on the rate of convergence of the core of an economy has recently been obtained (B. Grodal **[1974]**) for the case in which the agents' characteristics belong to a compact subset Q of $G \times P$. For a finite set A, d^A denotes the metric defined on the set of functions from A to R^l by $d^A(x, y) = \mathrm{Max}_{a \in A} \| x(a) - y(a) \|$, and $\delta^A(X, Y)$ denotes the associated Hausdorff distance of two compact sets X, Y of functions from A to R^l. In the statement of the theorem, $\mathcal{M}_Q$ denotes the set of characteristic distributions on Q with the topology of weak convergence.

THEOREM. *If Q is a compact subset of $G \times P$, and μ is a regular characteristic distribution on Q, then there are a neighborhood V of μ in $\mathcal{M}_Q$, and a real number k such that for every economy $\mathscr{E}$ with a finite set A of agents, and whose characteristic distribution belongs to V,*

$$\delta^A [C(\mathscr{E}), W(\mathscr{E})] \leqq k/\#A.$$

Thus if $(\mathscr{E}_n)$ is a competitive sequence of economies on Q, and if the limit characteristic distribution is regular, then $\delta^{A_n} [C(\mathscr{E}_n), W(\mathscr{E}_n)]$ tends to 0 at least as fast as the inverse of the number of agents.

The basic reference for this section is W. Hildenbrand **[1974]**.

The analysis of Walras equilibria, of the core, and of their relationship has yielded valuable insights into the role of prices in an economy. But possibly of greater importance has been the recognition that the techniques used in that analysis are indispensable for the mathematical study of social systems: algebraic topology for the test of existence that mathematical models of social equilibrium must pass; differential topology for the more demanding tests of discreteness, and of continuity for the set of equilibria; combinatorial techniques for the computation of equilibria; and measure theory for the study of large sets of small agents.

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